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infinite is greater than another infinite, or one eternal than another eternal; which,' he says, 'is absurd.' This demonstration is like his, who from this, that the number of even numbers is infinite, would conclude that there are as many even numbers as there are numbers simply, that is to say, the even numbers are as many as all the even and odd together. They which in this manner take away eternity from the world, do they not by the same means take away eternity from the creator of the world? . . . And the men that reason thus absurdly are not idiots, but, which makes this absurdity unpardonable, geometricians, and such as take upon them to be judges, impertinent, but severe judges of other men's demonstrations."

The reference to odd and even numbers doubtless arose from his contact with Galilean thought. While sojourning on the Continent, he had gone to see Galileo, then a prisoner. Hobbes thought he had effected the duplication of the cube and the squaring of the circle. On this matter he became involved in a heated controversy with the algebraist, John Wallis. The aged Hobbes was no match against young Wallis on mathematical questions. When the mathematical works of Wallis were being brought out, Wallis refused to allow his controversial matter against Hobbes to be incorporated in them.[1] Whether the whole is greater than a part was an issue touched upon during this dispute. Hobbes said to Wallis: "All this arguing of infinities is but the ambition of school boys." It cannot be said that Hobbes made any real contribution to a deeper understanding of the "Achilles" or any of Zeno's other arguments on motion. His objection to the dictum, "whatever may be divided into parts infinite in number, the same is infinite," is no new contribution; Aristotle had advanced that far. How Achilles caught the turtle is beyond comprehension through our sensual imagination; Hobbes nowhere explains this inability. However, he does touch upon the concept of a limit in his controversy with Wallis. Hobbes charged that some of the principles of the professors are "void of sense"; one of those principles being, "that a quantity may grow less and less eternally, so as at last to be equal to another quantity; or, which is all one, that there is a last in eternity."[2]

A GENERAL FORMULA FOR THE VALUATION OF SECURITIES.[3]

By JAMES W. GLOVER, University of Michigan.

The object of this paper is to derive a formula for the valuation of a very general type of securities. The security is redeemed in r equal installments at intervals of t years, the first redemption being made after f years. The annual rate of dividend is g payable in m installments, and the security is purchased to realize the investor a nominal rate of interest j with frequency of conversion m.

[1] A full account of the controversy between Hobbes and Wallis is given in Croom Robertson's *Hobbes*, pp. 167–185.

[2] *The English Works of Thomas Hobbes*, Vol. 7, p. 186.

[3] Read before the Chicago Section of the American Mathematical Society, April, 1912. Those unfamiliar with the notation and functions employed in the theory of compound interest may consult *Text-Book of the Institute of Actuaries*, Part I, by Ralph Todhunter; *The Mathematical Theory of Investment*, by Ernest B. Skinner; *Bulletin of the Department of Agriculture*, No. 136, on Highway Bonds, by Laurence I. Hewes and James W. Glover.

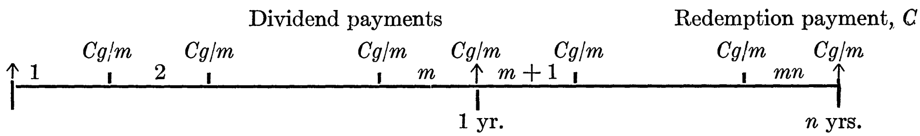
In order to derive this formula we first consider the simple case of a bond redeemed in one payment when the annual rate of dividend is g, payable in m installments, each equal to g/m, and the valuation is made at a nominal rate of interest j with frequency of conversion m.

Let C be the amount to be redeemed after n years.

Let g be the annual rate of dividend, payable in m equal installments g/m, per unit of the redemption fund C.

Let j be the nominal rate of interest, with frequency of conversion m, to be employed in the valuation.

Let A be the value of, or bid upon, the security.



Referring to the figure we see that the value of the security consists of two parts:

1. A series of dividend payments of Cg/m at the end of each mth part of the year for n years, and

2. The sum C to be redeemed at the end of n years.

The first part may be regarded as an immediate annuity-certain with payments of Cg/m per interval and running for mn intervals. Since interest is at the rate j/m per interval, the present value of this annuity is

$$a_{\overline{mn}|} \cdot Cg/m = \frac{1 - v^{mn}}{j/m} \cdot Cg/m = C(1 - v^{mn})g/j, \quad \text{where} \quad v = 1/(1 + j/m).$$

The present value of C, due in n years, or at the end of mn intervals, is Cv^{mn}, hence the present value of the security is

$$A = Cv^{mn} + (g/j)(C - Cv^{mn}), \tag{1}$$

where the function v^{mn} is to be taken at the rate j/m.

We are now prepared to consider the more general problem of valuing a security of the following nature:

1. The security is redeemed in r equal installments.

2. The first redemption payment is made at the end of f years.

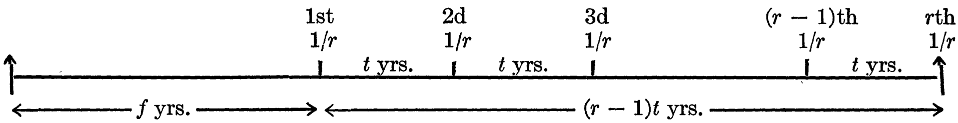
3. The remaining $(r - 1)$ redemption payments are made at intervals of t years.

4. The annual rate of dividend is g and dividends are paid in equal installments at the ends of the m equal intervals into which the year is divided.

5. The security is valued at the nominal rate $j_{(m)}$.

We proceed to find the present value A of a security of the above type whose total redemption fund is unity. Since the unit fund is redeemed in r equal installments each one will be $1/r$; the following figure illustrates the nature of the security.

Redemption payments.



It is evident that the total unit loan may be regarded as made up of r separate loans, each of $1/r$, and to determine its present value we need only to find the values of these several partial loans and add them together. For this purpose we set forth the following scheme:

No. of Loan.	Present Value.	Due or Matures in Years.	Amount Maturing per Unit of Total Sum to be Redeemed.
1	A_1	$n_1 = f$	$C_1 = 1/r$
2	A_2	$n_2 = f + t$	$C_2 = 1/r$
3	A_3	$n_3 = f + 2t$	$C_3 = 1/r$
.	.	. .	.
.	.	. .	.
.	.	. .	.
r	A_r	$n_r = f + (r - 1)t$	$C_r = 1/r$

Employing formula (1) for each one of these loans we have the present values as follows:

$$A_1 = \frac{1}{r}\, v^{mf} + (g/j)\left[\frac{1}{r} - \frac{1}{r}\, v^{mf}\right],$$

$$A_2 = \frac{1}{r}\, v^{m(f+t)} + (g/j)\left[\frac{1}{r} - \frac{1}{r}\, v^{m(f+t)}\right],$$

$$A_3 = \frac{1}{r}\, v^{m(f+2t)} + (g/j)\left[\frac{1}{r} - \frac{1}{r}\, v^{m(f+2t)}\right],$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$A_r = \frac{1}{r}\, v^{m(f+\overline{r-1}t)} + (g/j)\left[\frac{1}{r} - \frac{1}{r}\, v^{m(f+\overline{r-1}t)}\right],$$

all being at rate j/m.

Adding the left-hand column we have A, the present value per unit of the total sum to be redeemed. The first column on the right is a geometric series whose sum is $v^{mf}(1 - v^{mtr})/(1 - v^{mt})$, apart from the factor $1/r$, hence

$$A = \frac{1}{r} \cdot v^{mf} \cdot \frac{1 - v^{mtr}}{1 - v^{mt}} + (g/j)\left[1 - \frac{1}{r} \cdot v^{mf} \cdot \frac{1 - v^{mtr}}{1 - v^{mt}}\right].$$

For convenience, letting

$$z = \frac{1}{r} \cdot v^{mf} \cdot \frac{1 - v^{mtr}}{1 - v^{mt}},$$

the preceding formula may be written

$$A = z + (g/j)(1 - z),$$

whence the *premium* $k = A - 1$ takes the form

$$k = (1 - z)(g - j)/j. \tag{2}$$

Since z is the present value of a number of quantities whose sum is unity, $1-z$ must be positive, and formula (2) shows that the premium k is positive or negative according as the rate of dividend g is greater or less than the rate of interest j desired to be realized by the investor. When k is positive the security is said to be bought at a *premium*, when negative, at a *discount*.

The formula for the premium k on the unit loan expressed in terms of the present value or v-function is:

$$k = \left[1 - \frac{v^{mf}}{r} \cdot \frac{1 - v^{mtr}}{1 - v^{mt}}\right](g - j)/j, \qquad \text{at rate } j/m. \tag{3}$$

The most frequent case in practice is when $m = 2$. Formula (3) then becomes

$$k = \left[1 - \frac{v^{2f}}{r} \cdot \frac{1 - v^{2tr}}{1 - v^{2t}}\right](g - j)/j, \qquad \text{at rate } j/2. \tag{4}$$

These formulas may be modified somewhat since

$$z = \frac{v^{mf}}{r} \cdot \frac{1 - v^{mtr}}{1 - v^{mt}} = \frac{v^{mf}}{r} \cdot \frac{\dfrac{1 - v^{mtr}}{j/m}}{\dfrac{1 - v^{mt}}{j/m}} = \frac{v^{mf}}{r} \cdot \frac{a_{\overline{mtr}|}}{a_{\overline{mt}|}},$$

where $a_{\overline{n}|}$ is the present value of an immediate annuity-certain and one of the usually tabulated interest functions. We have then

$$k = [1 - v^{mf} a_{\overline{mtr}|}/r a_{\overline{mt}|}](g - j)/j, \qquad \text{at rate } j/m, \tag{5}$$

and in the special case when $m = 2$,

$$k = [1 - v^{2f} a_{\overline{2tr}|}/r a_{\overline{mt}|}](g - j)/j, \qquad \text{at rate } j/2. \tag{6}$$

It may be found desirable to express the value of the premium k in terms of the function $a_{\overline{n}|}$ and this can be accomplished as follows:

$$z = \frac{v^{mf}}{r} \cdot \frac{1 - v^{mtr}}{1 - v^{mt}} = \frac{1}{r} \cdot \frac{v^{mf} - v^{m(f+tr)}}{1 - v^{mt}}$$

$$= \frac{1}{r} \cdot \frac{\dfrac{1 - v^{m(f+tr)}}{j/m} - \dfrac{1 - v^{mf}}{j/m}}{\dfrac{1 - v^{mt}}{j/m}} = \frac{1}{r} \cdot \frac{a_{\overline{m(f+tr)}|} - a_{\overline{mf}|}}{a_{\overline{mt}|}},$$

where all the annuities are to be taken at rate j/m.

This leads to the formula

$$(7) \qquad k = \left[1 - \frac{a_{\overline{m(f+tr)}|} - a_{\overline{mf}|}}{r a_{\overline{mt}|}}\right](g - j)/j, \qquad \text{at rate } j/m,$$

and for the important practical case $m = 2$,

$$(8) \qquad k = \left[1 - \frac{a_{\overline{2(f+tr)}|} - a_{\overline{2f}|}}{r a_{\overline{2t}|}}\right](g - j)/j, \qquad \text{at rate } j/2.$$

Believing that the readers of the MONTHLY may be interested in the operation of a bond loan of this character we give the following example.

What is the premium on \$100,000 highway bonds, interest 5% payable semiannually, dated January 1, 1914, maturing \$50,000 January 1, 1917, and \$50,000 January 1, 1919, to net the purchaser 4 per cent. compounded semiannually?

Here $f = 3$, $r = 2$, $t = 2$, $m = 2$, $g = .05$, $j = .04$, hence $m(f + tr) = 14$, $mf = 6$, $mt = 4$. Consulting a table[1] of annuities, $a_{\overline{n}|}$, with 2 per cent. as the rate of interest, and employing formula (8), the numerical work may be outlined as follows:

$$\begin{array}{rll}
a_{\overline{14}|} = & 12.10624877 & (1) \\
a_{\overline{6}|} = & 5.60143089 & (2) \\
a_{\overline{14}|} - a_{\overline{6}|} = & 6.50481788 & (3) \\
(3) \div 2 = & 3.25240894 & (4) \\
a_{\overline{4}|} = & 3.80772870 & (5) \\
(4) \div (5) = & .85415984 & (6) \\
\text{Complement of } (6) = 1 - (6) = & .14584016 & (7) = \text{first factor} \\
(.05 - .04)/.04 = & .25 & (8) = \text{second factor} \\
k = (7) \times (8) = & .03646004 &
\end{array}$$

The bid on one dollar is 1.03646004, hence the bid on the entire issue is \$103,646.004. The progress of the loan is indicated in the following schedule.

(1)	(2)	(3)	(4)	(5)	(6)	(7)
Interval.	Year.	Book Value or Principal at Beginning of Interval.	Semiannual Interest of 2%.	Semiannual Dividend of $2\frac{1}{2}$% on Bonds.	Amortization of Premium at End of Interval.	Redemption Payment at End of Interval.
1	$\frac{1}{2}$	\$103,646.00	\$2,072.92	\$2,500.00	\$427.08	0.00
2	1	103,218.92	2,064.38	2,500.00	435.62	0.00
3	$1\frac{1}{2}$	102,783.30	2,055.67	2,500.00	444.33	0.00
4	2	102,338.97	2,046.78	2,500.00	453.22	0.00
5	$2\frac{1}{2}$	101,885.75	2,037.72	2,500.00	462.28	0.00
6	3	101,423.47	2,028.47	2,500.00	471.53	\$50,000.00
7	$3\frac{1}{2}$	50,951.94	1,019.04	1,250.00	230.96	0.00
8	4	50,720.98	1,014.42	1,250.00	235.58	0.00
9	$4\frac{1}{2}$	50,485.40	1,009.71	1,250.00	240.29	0.00
10	5	50,245.11	1,004.89	1,250.00	245.11	50,000.00
Totals		817,699.84	16,354.00	20,000.00	3,646.00	100,000.00

[1] Compound Interest and Annuity Calculations with Tables, W. M. J. Werker.

By adding the several columns in this schedule various checks are obtained which are too evident to need comment. Attention is called to the fact, however, that it is not necessary to construct the whole schedule in order to determine an item in a given row and column. For example the fifth item in column three, \$101,885.75, representing the present value of the outstanding security at the beginning of the fifth interval, can be calculated directly by making the proper substitutions in formula (8). When this is known the other items in the same row can be determined at once.

There are several special cases of formula (7) which deserve mention. The most common type of serial bond bears semiannual dividends and is redeemed in n equal *annual* installments, the first of which is paid at the end of the first year. In this case $f = t = 1$, $r = n$, $m = 2$, and

$$k = \left[1 - \frac{a_{\overline{2n+2}|} - a_{\overline{2}|}}{n a_{\overline{2}|}}\right](g - j)/j, \qquad \text{at rate } j/2. \tag{9}$$

Since

$$a_{\overline{2n+2}|} = a_{\overline{2}|} + v^2 a_{\overline{2n}|} \quad \text{and} \quad v^2/a_{\overline{2}|} = 1/s_{\overline{2}|},$$

formula (9) may be written

$$k = \left[1 - \frac{a_{\overline{2n}|}}{n s_{\overline{2}|}}\right](g - j)/j, \qquad \text{at rate } j/2. \tag{10}$$

When the serial bond is like the preceding except that dividends are payable and interest is convertible annually we have $f = t = 1$, $r = n$, $m = 1$, and

$$k = \left[1 - \frac{a_{\overline{n+1}|} - a_{\overline{1}|}}{n a_{\overline{1}|}}\right](g - j)/j, \qquad \text{at rate } j/1.$$

In this case the nominal rate $j_{(1)}$ equals the effective rate i and, since $a_{\overline{n+1}|} = a_{\overline{1}|}\,(1 + a_{\overline{n}|})$, the formula reduces to

$$k = \left[1 - \frac{a_{\overline{n}|}}{n}\right](g - i)/i. \tag{11}$$

When the bond is redeemed in a single installment at the end of its term, say n years, we have, $f = n$, $r = 1$, and formula (7) reduces to

$$k = \left[1 - \frac{a_{\overline{m(n+t)}|} - a_{\overline{mn}|}}{a_{\overline{mt}|}}\right](g - j)/j,$$

but since

$$a_{\overline{mn+mt}|} = a_{\overline{mn}|} + v^{mn} \cdot a_{\overline{mt}|},$$

the formula for the premium may be written

$$k = \left[\frac{1 - v^{mn}}{j/m}\right](g - j)/m, \qquad \text{at rate } j/m$$

or finally,

$$k = a_{\overline{mn}|}\,(g - j)/m, \qquad \text{at rate } j/m. \tag{12}$$

When, as is usually the case, dividends on the bond are paid semiannually and it is valued to net the purchaser a nominal rate j convertible twice a year, formula (12) becomes

$$k = a_{\overline{2n}|}\,(g - j)/2, \qquad \text{at rate } j/2. \tag{13}$$

In the simplest of all cases, dividends payable annually at rate g, interest compounded annually at the effective rate i, and the bond maturing in one sum at the end of n years, (12) reduces to the well-known form

$$k = a_{\overline{n}|}\,(g - i). \tag{14}$$

This formula admits of a simple interpretation because it states that the premium per unit of the sum to be redeemed is equal to the present value of an annuity whose annual rent is equal to the excess of the rate of dividend over the rate of interest desired to be realized by the purchaser. I may add that practically all the formulas in this paper admit of a direct interpretation. The interpretation of the final formula usually suggests a simple method of deriving it by general considerations and always throws a great deal of light upon the nature of the problem. It is not my purpose, however, to enter at this time upon the subject of interpretation of interest formulas.

QUESTIONS AND DISCUSSIONS.

Edited by U. G. Mitchell.

NEW QUESTIONS.

24. The following facts are significant:

(1) The New England Association of Mathematics Teachers has appointed a committee "to investigate the current criticisms of high school mathematics."

(2) A committee of the Council of the American Mathematical Society has under consideration the question "whether any action is desirable on the part of the Society in the matter of the movement against mathematics in the schools."

(3) At the recent meeting in Cincinnati of the National Education Association an iconoclastic discussion on the topic: "Can algebra and geometry be reorganized so as to justify their retention for high school pupils not likely to enter technical schools?" aroused approbation and applause. An outline of the remarks by one of the speakers will be printed in this column next month.

In view of these facts what should be done by those who believe in the value of mathematics as a general high school study?

REPLIES.

9. What is the present state of experience with coördinated courses in high school mathematics? What contribution does this promise to the development of mathematics teaching in high schools? What about the corresponding matters in college mathematics? (*Note.*—An individual correspondent need not answer all the questions in number 9; it is sufficient if he answers only one.)

Reply by Roy Cumins, Columbia University, N. Y.

At present there exists in the United States a decided movement toward breaking down the barriers that have hitherto kept separate the various branches